

Short Introduction to Space-Rocket Dynamics

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1 The simplified one-dimensional rocket equation

This note aims at shortly explaining some of the principles governing the motion of a pure rocket. We shall exclude externally powered space vehicles, though realizable in the future. One assumes that the reader knows the basic concepts of classical physics and some elements of calculus¹.

We shall use two frames of reference: one is the spacecraft frame (or **SF**, for short), the other one is an inertial frame (or **IF**, for short) with respect to which the rocket body motion is described. Let us begin by carrying out the expression of the thrust a rocket produces in the *free-field* environment. It is straightforward to write the momentum conservation law:

$$m d\mathbf{V} = -\mathbf{U} dm_p \quad (1)$$

where m is the vehicle's mass, \mathbf{V} is its velocity in **IF**, and dm_p is the (propellant) mass ejected at velocity \mathbf{U} in **SF**. In the *1-dimensional case* (1D), we note that

$$\text{sign}(\mathbf{U} \cdot d\mathbf{V}) = -1 \quad (1a)$$

anyway (namely, from a formal viewpoint, a decelerating rocket is equivalent to an accelerating one). By assuming that the vehicle has no mass loss but the propellant

$$dm(t) = -dm_p(t) \quad (1b)$$

and setting $v \equiv |\mathbf{V}|$, $u \equiv |\mathbf{U}|$, then we can write

$$m(t) dv(t) = -u dm(t) \quad (2)$$

¹ In the final section of this paper, we shall mention a few items of geometric algebra, essentially for showing some of its calculation features and inviting the reader to deepen that topic.

Equation 2 is the rocket equation holding in the field-free 1D case under the assumption (1b). In general, \dot{m}_p represents the flow-rate of propellant (from the tankage to the engine) accelerated up to u (which can be set constant in the current context without loss of generality). Whatever the function $\dot{m}_p(t)$ may be, from the previous equations one gets the vector thrust:

$$\mathbf{T} \equiv m\dot{\mathbf{V}} = -\dot{m}_p\mathbf{U} = \dot{m}\mathbf{U} \quad (3)$$

Integrating eqn.-2, one obtains the rocket speed

$$v(t) = v_0 - u \int_{m_0}^{m(t)} \frac{dx}{x}$$

which can be put in a simpler form as follows:

$$\Delta v(t) = u \ln \left[\frac{m_0}{m(t)} \right] = u \ln R(t) \quad (4)$$

where $R(t)$ is the so-called propulsion mass ratio at time t . The design value of this function is its value after the completion of all maneuvers or, simply, R_f .

Equation 4 is the rocket equation in its simplest form. It shows the importance of getting engine jet speed values as high as possible compatibly with design constraints.

One should not forget that eqn.-4 holds in the ideal field-free environment or in the so-called impulsive approximation. In every real case, one has to take other field(s) into account and the resulting equation is more complicate. As a point of fact, often, neglecting such fields could cause big design mistakes.

Note: as mentioned above, one ideal assumption is that *all* propellant from tankage to engine may be accelerated up to the speed u . That's not possible in real cases: only a fraction η_m can be u -ejected. Equation 4 has to modified into

$$\Delta v(t) = \eta_m u \ln \left[\frac{m_0}{m(t)} \right] = u_e \ln R(t) \quad (4a)$$

where u_e represents the *effective* exhaust speed. Thus, with the same propulsion ratio, the vehicle's final speed is lower.

Going back to eqn.-4, it tells us that the spacecraft speed increment over a time interval $[t_1, t_2]$ is: (a) dependent on the value m_2/m_1 , no matter how the function $\dot{m}_p(t)$ may behave in this interval, (b) independent of the choice of $[t_1, t_2]$ provided that the propulsion ratio does not change, (c) independent of the vehicle speed at t_1 . In general, properties (a-b) do not hold if the spacecraft undergoes some external field such as gravity or drag. However, if the maneuver is impulsive² (or quasi-impulsive, in practice) eqn.-4 still holds.² Anyway, the propulsion mass ratio keeps its meaning. Also, note that the actual u is an effective value mainly related to the physical processes utilized for producing thrust; a case in point, the exhaust speed of a chemical engine operating in atmosphere depends also on the external pressure.

the so-called impulsive approximation consists of assuming a vanishing burning time and an unbounded vector acceleration, but with a finite product. As a result, the spacecraft position does not change while its velocity passes from \mathbf{v} to $\mathbf{v} + \Delta\mathbf{v}$.

2 Energetics

2.1 the rocket body

In the current context of field-free 1D path, the overall energy consists of kinetic energy only. It varies according to

$$\dot{E}_R = \frac{d}{dt}(\frac{1}{2}m v^2) = \mathbf{T} \cdot \mathbf{V} - \frac{1}{2}v^2 \dot{m}_p = \mathbf{T} \cdot \mathbf{V} - \frac{a}{u} E_R \quad (5)$$

where we have used eqns. 2-3 and $T = ma$. Since the infinitesimal work done by the thrust is equal to $dL_R = \mathbf{T} \cdot \mathbf{V} dt$, multiplying eqn.-5 by dt gives the following inequality:

$$dE_R \leq dL_R \quad (6)$$

This result was expected because the energy associated to the thrust, acting on the rocket, is shared between the body and the propellant, which is subsequently exhausted away. In point of fact, the supplementary term in eqn.-5 is just the kinetic power - in IF - of the propellant *before* its ejection.

2.2 the propellant

In IF, the propellant's energy variation is given by:

$$dE_p = \frac{1}{2} (\mathbf{V} + \mathbf{U})^2 dm_p \quad (7)$$

From eqns. 5 and 7 one immediately gets

$$dE_R + dE_p = \frac{1}{2} u^2 dm_p \quad (8)$$

The right-hand side of eqn.-8 is nothing else than the (chemical, electromagnetic, nuclear, etc.) onboard energy necessary to accelerate dm_p in SF. Since the following equality holds

$$dL_R + dL_p = dE_R + dE_p \quad (9)$$

then, the elementary work done on the propellant is given by

$$dL_p = (\mathbf{V} + \frac{1}{2}\mathbf{U}) \cdot \mathbf{U} dm_p \leq dE_p \quad (9a)$$

Inequality-9a is the counterpart of ineq.-6 due to the energy conservation (9).

2.3 the rocket in a gravitational field

Let us now extend the above considerations to a rocket spacecraft moving in a central gravitational field. As before, we shall ignore the motion about the vehicle's barycenter. We can write $E_R = \frac{1}{2}m v^2 + m \phi$ where ϕ is the potential energy per unit mass at distance r from the central body. Note that the propellant has a potential energy as well; therefore, in the time interval dt , there are four variations of energy, but their sum has to equal the overall energy released inside the rocket³. The following equation follows:

$$d(\frac{1}{2}m v^2 + m\phi) + (\frac{1}{2}|\mathbf{V} + \mathbf{U}|^2 + \phi) dm_p = \frac{1}{2}u^2 dm_p \quad (10)$$

With respect to the field-free case, a decrease of the vehicle's kinetic energy may be partially "mitigated" by the increase of its gravitational potential. This does not take place when a dissipative force, e.g. the aerodynamic drag, brakes the vehicle⁴. Expanding eqn.-10 and inserting eqn.-1b results in

³ In the current framework, we assumed again that the propellant is given all such energy.

⁴ In the case of a multi-staged rocket lifting off, the ignition of the second stage is delayed appropriately since drag depends on atmosphere density and rocket speed.

$$m d(\frac{1}{2}v^2 + \phi) = \mathbf{V} \cdot \mathbf{U} dm \quad (10a)$$

Differentiating the vehicle energy with respect to time, one obtains

$$\frac{dE_R}{dt} = \frac{d}{dt}(\frac{1}{2}m v^2 + m\phi) = \left(v\dot{v} + \frac{d\phi}{dr}\dot{r} \right) m + (\frac{1}{2}v^2 + \phi) \dot{m} \quad (11)$$

or, in terms of thrust,

$$\dot{E}_R + \frac{a}{u}E_R - \mathbf{T} \cdot \mathbf{V} = 0 \quad (11a)$$

that does not differ from eqn.-5 formally. However, the initial conditions do. One should note that eqn.-11a has been obtained without using any restriction on the rocket trajectory dimensions.

Equation-10a already contains the effect of gravity on the spacecraft speed. Nevertheless, in order to show this one more explicitly, we shall deal with the two-dimensional equations of motion of a spacecraft thrust *tangentially* in a Keplerian field. The polar form is appropriate for analytical purposes:

$$\begin{aligned} \ddot{r} &= a\dot{r}/v - \mu/r^2 + r\dot{\psi} \\ \ddot{\psi} &= a\dot{\psi}/v - 2\dot{r}\dot{\psi}/r \end{aligned} \quad (12)$$

where $a = T/m$ again. Equations 12 can be carried out strictly by the Euler-Lagrange equations with non-conservative terms due to the thrust field. Then, by along-track projecting the total acceleration, one has:

$$\mathbf{V} \cdot \dot{\mathbf{V}} = \mathbf{V} \cdot (\mathbf{T} / m + \mathbf{g}) \quad (13)$$

that, applied to the current case, gives

$$v\dot{v} = a v - (\mu / r^2)\dot{r} \quad (13a)$$

This allows one to express the time rate of the vehicle speed as function of the thrust and gravity accelerations. Equation 13a can be also obtained from equation 10a with $\mathbf{U} // - \mathbf{V}$. It represents the rocket equation in *differential form*. For the current aims, we need its integral form. Thus, integrating over the interval $[\rho, \tau]$ of propulsion results in the following equation

$$\Delta v(\tau) = \int_{\rho}^{\tau} a(t) dt - \mu \int_{\rho}^{\tau} (\cos \varphi / r^2) dt \quad (14)$$

where φ is the angle from \mathbf{R} to \mathbf{V} . In principle, the gravity speed losses cannot be calculated a-priori because one should know the evolution of the spacecraft state $[\mathbf{R}, \mathbf{V}]$. However, in the case of *low* thrust acceleration, the two dimensional trajectory, expressed by eqns.-12 with a circular departure orbit, is a spiral for most of the propulsion time. The second integral in eqn.14 can be developed to give:

$$\Delta v^{(g)} = -a_o t \left(2 + \frac{a_o t}{u_e} \right) + O((a_o t)^3), \quad t \leq t_{esc} \quad (14a)$$

where the escape time is the time at which $E_R = 0$. Note the factor 2 in eqn.-14a. Spiralling with $\mathbf{a} \cdot \mathbf{V} > 0$, but with a *low* acceleration value, the vehicle increases its distance from the central body, but its (mean orbital) speed decreases significantly; however, eventually, it reaches a point where its energy is zero (strictly, the minimum of speed occurs shortly before the energy vanishes). It can be shown by numerical integration of the eqns.-12 that, if burning continues past the escape, energy becomes positive and speed raises monotonically. If the thrust acceleration is of the order of the local g , then expression 14a does not hold.

Now, we shall use a very particular case of eqns.-12, i.e. a rocket rising vertically in a (practically-)constant gravity g (over the considered range of altitude). Thus, it is a simple task to carry out:

$$\Delta v(\tau) = u_e \ln R(\tau) - g\tau \quad (15)$$

One may think of lessening thrust acceleration and increasing burning time, but keeping $R(\tau)$ at some fixed value. However, since one cannot control g , using longer propulsion time augments the gravity losses. In general, the effect of gravity on speed is expressed by

$$\Delta v^{(g)}(\tau) = \int_0^\tau \mathbf{g} \cdot (\mathbf{V} / v) dt \quad (16)$$

where one can see that the along-track component of the gravitational acceleration is responsible for speed losses or gains. However, one should be careful even about "gaining". A case in point is a rocket descending vertically toward the surface of a (non-rotating) planet and landing softly. Now, the goal is to reduce the final speed ($v_f \cong 0$). If the burning time is sufficiently brief, then

we shall show that one should apply the speed change at low altitude. In contrast, if τ is long, then from eqn.-13 one has

$$-v_o = u_e \ln \frac{m_o}{m_o - m_p} - g \tau \implies m_p = \left[1 - \exp \left(-\frac{|v_o| + g\tau}{u_e} \right) \right] m_o \quad (17)$$

because of $v_o < 0$. It is evident from eqn.-17 that a sufficiently long burning time, caused by low acceleration, entails a propellant consumption that, for a given spacecraft, exceeds the capacity of its tanks; as a result, the vehicle will have a residual speed that would not allow a soft landing. Alternatively, if one designs a soft-landing vehicle by low thrust acceleration, the payload would be consequently low.

The so-called gravity-losses can be encompassed in the more general concept of finite-burn losses, which holds in the free-field case too. This topic is beyond the aims of this paper.

2.4 simple principles for maneuvering

1. At the end of Section-1, we gave the definition of impulsive approximation. Algorithms containing such approximation are enormously simplified, while they retain a certain physical meaning. Here, we shall use it for explaining the so-called high-altitude burning effect in changing orbit by rocket engines. Let us envisage a rocket passing by free-fall from a point P_1 to a higher point P_2 in a gravitational field. At P_2 , an impulsive change of velocity takes place: $\mathbf{V}_2 \rightarrow \mathbf{V}_2 + \delta\mathbf{V}$, assuming $\delta\mathbf{V} \ll \mathbf{V}_2$ for simplicity. Before the impulse, the conservation of energy in free-fall entails

$$\frac{1}{2}v_2^2 = \frac{1}{2}v_1^2 - (\phi_2 - \phi_1) \quad (18)$$

whereas, after the increment, the energy per unit mass takes on $E_{m2}^+ = \frac{1}{2}(v_2 + \delta v)^2 + \phi_{r2}$. Inserting it into eqn.-18 results in

$$E_{m2}^+ = \frac{1}{2}(v_1 + \delta v)^2 + \phi_1 - (v_1 - v_2) \delta v \quad (19)$$

Now, let us consider the reverse situation: the impulse is applied at P_1 and the rocket rises up to some P_2' . After the impulse, one has $E_{m1}^+ = \frac{1}{2}(v_1 + \delta v)^2 + \phi_1$. The energy difference just past the respective

impulses is given by

$$E_{m1}^+ - E_{m2}^+ = (v_1 - v_2) \delta v > 0 \quad (20)$$

The inequality follows because $v_1 > v_2$ by definition of the two points. In words, equation 20 tells us that giving an impulsive change of velocity at higher altitude in a gravity field provokes a loss of specific energy proportional to the given speed change. This one is the *high-altitude burning* effect, which has important consequences in orbital manoeuvres. Here are two simple examples: (1) a rocket-vehicle, parked on a circular orbit, has to be accelerated (possibly by a single impulse) for escaping, that is $E_m^+ \geq 0$. By eqn.-20, propellant can be lessened if the parking altitude is low (compatibly with mission constraints); (2) if the parking or, more generally, the departure orbit is elliptical, then the burn should occur at the perifocus.

2. Real maneuvers for a general transfer flight could depart significantly from the above simple scheme. In particular, in transfers for which time is critical, the related optimal strategies may be much different from the ideal minimum-propellant solution. Let us mention some simple, though important, principles for best changing the spacecraft's orbital energy and/or angular momentum. Suppose to want to change spacecraft energy:

$$\delta E_R = \delta \left(\frac{1}{2} m \mathbf{V} \cdot \mathbf{V} + m \phi \right) = m \mathbf{V} \cdot \delta \mathbf{V} + \frac{1}{2} v^2 \delta m + \delta (m \phi) \quad (21)$$

δE_R can be maximized/minimized by $\delta \mathbf{V}$ parallel or antiparallel to \mathbf{V} . Terms proportional to δm are negligible for rocket engines with high exhaust speed. With regard to the energy per unit mass, one gets the same conclusion regardless any propellant ejection:

$$\delta E = \mathbf{V} \cdot \delta \mathbf{V} + \delta \phi \quad (22)$$

3. Now, let us consider the orbital angular momentum. We shall use **geometric algebra**⁵ for gaining both algebraic solution and geometric meaning. The angular momentum bivector is defined by

$$\Lambda = m \mathbf{R} \wedge \mathbf{V} \quad (23)$$

⁵ <http://www.clifford-algebras.org/>

[http://www.clifford.org/clf-
alg/](http://www.clifford.org/clf-
alg/)

<http://math.tntech.edu/rafal/cliff9/>

where \wedge denotes the outer product. Therefore,

$$\delta\Lambda = \delta(m\mathbf{R} \wedge \mathbf{V}) = \delta m \mathbf{R} \wedge \mathbf{V} + m \delta\mathbf{R} \wedge \mathbf{V} + m \mathbf{R} \wedge \delta\mathbf{V} \quad (24)$$

$\delta\Lambda$ attains its maximum/minimum value when $\delta\mathbf{V} \perp \mathbf{R}$ (the velocity change is the only control for a given δm). In the impulsive approximation ($\delta\mathbf{R} = 0$) and negligible ejected mass, $\max(\delta\Lambda) = m \mathbf{R} \wedge \delta\mathbf{V} = m \mathbf{R} \delta\mathbf{V}$ (here, *the geometric product* is denoted by *juxtaposition*), since the inner product $\mathbf{R} \cdot \delta\mathbf{V}$ vanishes. If the goal of the maneuver is to accomplish a given $\delta\Lambda$, then the velocity change immediately follows:

$$\delta\mathbf{V} = \frac{1}{m} \mathbf{R}^{-1} \delta\Lambda = \frac{1}{mr^2} \mathbf{R} \delta\Lambda \quad (24a)$$

(by a peculiar property of the geometric algebra). Considering the angular momentum bivector per unit mass $H \equiv \mathbf{R} \wedge \mathbf{V}$, one has:

$$\delta H = \mathbf{R} \wedge \delta\mathbf{V} + \delta\mathbf{R} \wedge \mathbf{V} \quad (25)$$

In the impulsive approximation, maximizing/minimizing the H -change results in

$$\delta\mathbf{V} = \mathbf{R}^{-1} \Delta H = \frac{1}{r^2} \mathbf{R} \delta H \quad (25a)$$

Note that eqn.-25a is always exact, independently of how much mass is exhausted by the engines. Since we are in the impulsive approximation, $\frac{\Delta\mathbf{V}}{|\Delta\mathbf{V}|} = \frac{\mathbf{T}}{|\mathbf{T}|}$, where \mathbf{T} is the thrust vector. Conversely, eqn.-25a may be read as the minimum of all speed changes accomplishing a given $H_f - H_0$; any other thrust direction increases the propellant consumption. This can be easily visualized geometrically by given initial and final orientations of the H -plane in space.

3. Some remarks are in order.

- (a) The above geometric-algebra equations are *coordinate-free*. One could need to solve the above problem in a particular reference frame with conventional orthonormal basis

$$\{\mathbf{e}_{(i)}, i = 1..3\}, g_{ij} = \mathbf{e}_{(i)} \cdot \mathbf{e}_{(j)} = \delta_{ij} \quad (26)$$

Then, one can write the position vector as $\mathbf{R} = x^k \mathbf{e}_{(k)}$.

- (b) The bivector δH cannot be completely arbitrary because of the constraint $\Delta \mathbf{R} = \mathbf{0}$. This one entails that the components of the angular momentum change

$$\delta H = \mathbf{R} \wedge \mathbf{W} = h^{i,j} \mathbf{e}_{(i)} \wedge \mathbf{e}_{(j)} \equiv h^{i,j} \mathbf{e}_{i,j} \quad i < j \quad (27)$$

have to satisfy the following constraint

$$x^3 h^{1,2} - x^2 h^{1,3} + x^1 h^{2,3} = 0 \quad (27a)$$

Thus, in principle, there are only two degrees of freedom: the angle of rotation of H_f about \mathbf{R} and the area of H_f .

- (c) Inserting the expressions of \mathbf{R} and δH into eqn.-25a results in

$$r^2 \Delta \mathbf{V} = (-x^2 h^{1,2} - x^3 h^{1,3}) \mathbf{e}_{(1)} + (-x^3 h^{2,3} + x^1 h^{1,2}) \mathbf{e}_{(2)} + (x^1 h^{1,3} + x^2 h^{2,3}) \mathbf{e}_{(3)} \quad (28)$$

where $r^2 = \mathbf{R} \cdot \mathbf{R}$ and with eqn.-27a. The usual angular momentum vector (an axial vector), say, $\mathbf{H} = \mathbf{R} \times \mathbf{V}$ is tied to H via

$$\mathbf{H} = -I H \quad \Longleftrightarrow \quad H = I \mathbf{H} \quad (29)$$

where $I = \mathbf{e}_{(1)} \mathbf{e}_{(2)} \mathbf{e}_{(3)}$ ($I^2 = -1$ according to the present signature) is the *pseudoscalar* of the usual three-dimensional Euclidean space. In addition, by eqn.-29, it is straightforward to show

$$\mathbf{H} \cdot H = 0 \quad H \wedge \mathbf{H} = |\mathbf{H}|^2 I \quad (30)$$

namely, \mathbf{H} is along the positive normal to H .⁶

5. In the standard two-impulse Hohmann transfer between two arbitrary-radii co-planar circular orbits, the two impulses are performed at the perifocus and the apofocus of the transfer ellipse: both impulses are parallel to the local velocities *and* orthogonal to the respective vector positions. These features repeat in the three-impulse bi-elliptic transfer mode⁷. Thus, according to the above sub-sections 2-3, there is a simultaneous maximization/minimization of energy and angular momentum in both techniques. No surprise, then, that each transfer mode implies the *minimum propellant* in wide ranges of the ratio between the final and initial orbit radii.

⁶ in eqn.-30, the inner product between a vector and a bivector, in the order, is defined compliantly with the left-contraction operation.

⁷ the third impulse is antiparallel to the velocity

6. In the two-body Kepler problem, a general conic has eccentricity ε given by

$$\varepsilon^2 = 1 + 2 |\mathbf{H}|^2 E / \mu^2 \quad (31)$$

where, again, E is the orbit energy ($=E_R/m$). The orbit eccentricity and the apofocus-to-perifocus direction can be summarized by means of the eccentricity vector. It can be expressed via geometric algebra:

$$\boldsymbol{\varepsilon} = \mu^{-1} H \mathbf{V} - \mathbf{R} / r \quad (32)$$

An impulsive velocity change induces a 1st-order variation in $\boldsymbol{\varepsilon}$

$$\delta\boldsymbol{\varepsilon} = \mu^{-1} (\delta H \mathbf{V} + H \delta\mathbf{V}) \quad (33)$$

For a circular orbit, $H_c = \mathbf{R}_c \wedge \mathbf{V}_c = \mathbf{R}_c \mathbf{V}_c$ and eqn.-33 becomes

$$(\delta\boldsymbol{\varepsilon})_c = 2 \mu^{-1} (\mathbf{V}_c \cdot \delta\mathbf{V}) \mathbf{R}_c \quad (33a)$$

Equation 33a tells us that the maximum change of the eccentricity vector happens for a tangential impulse. In contrast, an orthogonal impulse produces only a 2nd-order change equal to $2 \mu^{-1} (\delta v)^2 \mathbf{R}_c$.